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A Personal Perspective on the Last Half-Century of Critical Phenomena

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Abstract One of the crowning achievements of mathematical, statistical physics over the past half century has been the discovery of the many aspects of structure of the critical point. It has been an exciting time and was only possible through the combined efforts of many excellent people. This article contains brief reviews of some of the parts in which I have been most interested and to which I have made some contributions.

Keywords Critical phenomena · Padé approximants · Renormalization group · Monte Carlo · Equation of state

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1 Introduction

This manuscript makes no pretense of being complete as is for example the book by Domb [30], or the review of Pelissetto and Vicari [49]. Rather it gives the perspective of one researcher who did make a significant contribution along the way to the modern theory of critical phenomena. Some of the material discussed here is treated at greater length in my book [9].

2 Classical Results

The year 1869 was marked by the discovery by Andrews [4] that there was a very special point for carbon dioxide at about 31°C and 73 atmospheres where the properties of the liquid and the gas became indistinguishable. He named that point the critical point. In the neighborhood of this point he found that carbon dioxide became opalescent, that is light is strongly scattered. The isothermal compressibility diverges. Before that time, scientists

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thought that there were permanent gases which could not be liquefied, no matter how much pressure was applied. In the subsequent years a wide variety of substances were found with similar properties. Of particular note was the discovery by P. Curie [28] that the ferromagnet iron also displayed a special point. This is the highest temperature in zero magnetic field at which it can remain permanently magnetized. Curie himself was struck by the parallelism between the density-temperature curves at constant pressure for carbon dioxide and his magnetization-temperature curves at constant magnetic field. In magnets, the special point is called the Curie point. Analogously, as the Curie point is approached the susceptibility diverges.

The first theoretical effort to explain his result was by Van der Waals [59]. He proposed the equation of state

$$p = \frac{kT}{v-b} - \frac{a}{v^2} \tag{1}$$

The critical point of his equation is

$$v_c = 3b, \qquad p_c = \frac{1}{27}\frac{a}{b^2}, \qquad kT_c = \frac{8}{27}\frac{a}{b}, \tag{2}$$

where a and b are constants depending on the material. For temperatures below T_c in this equation of state when pressure is plotted vs. volume there is a violation of the thermodynamic requirement that $\partial p/\partial V|_T \leq 0$. The solution is the famous Maxwell construction. Here two points are selected which are joined by a line of constant pressure, such that for the Van der Waals equation of state $\int_{V_1}^{V_2} p dV = 0$. This is the famous equal areas construction.

In some other approximate equations of state there is a region of negative specific heat at constant volume. C_V is a principal specific heat so this is a violation of thermodynamics. If we note the thermodynamic relations

$$\left.\frac{\partial A}{\partial T}\right|_V = -S, \qquad \left.\frac{\partial U}{\partial T}\right|_V = T\left.\frac{\partial S}{\partial T}\right|_V = C_V = -T\left.\frac{\partial^2 A}{\partial T^2}\right|_V, \tag{3}$$

where U is the internal energy, A is the free energy, and S is the entropy, then solution [10] to this problem is analogous to the Maxwell construction. The condition that no work be done is that $U(T_1, V) = U(T_2, V)$ and the condition that slopes of the Helmholtz free energy with respect to temperature be equal is $S(T_1, V) = S(T_2, V)$, or

$$\int_{T_1}^{T_2} \frac{dt}{t} \left.\frac{\partial U(t, V)}{\partial t}\right|_V = 0 \tag{4}$$

This solution is straight forward, but I haven't noticed it in the literature.

3 Critical Point Structure

In addition to finding the critical point, it is of interest to characterize its structure. This is commonly done by the use of the so called critical indices. In magnetic language the magnetic susceptibility diverges like $(T - T_c)^{-\gamma}$. The two particle correlation length ξ diverges like $(T - T_c)^{-\nu}$. The specific heat diverges like $(T - T_c)^{-\alpha}$ and the magnetization vanishes like $(T_c - T)^{\beta}$. γ' and α' denote the corresponding quantities below the critical temperature.

For $T = T_c$ $H \propto M^{\delta}$. Fisher [34] introduced a critical exponent η to measure the deviation of the two-point correlation from Ornstein-Zernike predictions. It is defined by

$$g(r) \propto r^{2-d-\eta}, \quad \text{as } r \to \infty, \qquad T = T_c \tag{5}$$

He also introduced ω^* as the anomalous dimension of the vacuum. Newman's [46] inequalities imply that $\omega^* \geq 0$. There were, during this time frame proven a number of rigorous inequalities. I list here some of the them which apply to the ferromagnetic Ising model, and of course a number of other models. From thermodynamics we get Rushbrooke's inequality [50] $\alpha' + 2\beta + \gamma' \geq 2$ and Griffith's thermodynamic inequality [40] $2 - \alpha \leq \beta(\delta + 1)$. Fisher's inequality [35], $\gamma \leq (2 - \eta)\nu$, Sokal's inequality[56], $\gamma \leq 2\nu$, Josephson's inequality [42] $d\nu' \geq 2 - \alpha'$, $d\nu \geq 2 - \alpha$ (proved by Sokal) [55] In addition Sokal showed that $d\nu' \geq \gamma' + 2\beta \geq 2 - \alpha'$ For the spin one-half Ising model, Simon [52] proved $\eta \leq 1$. Glimm and Jaffe [39], and Baker [5] proved that $\gamma \geq 1$, $\nu \geq 2/(d+1)$ and Aizenman [2, 3] proved for $d \geq 5$ that $\gamma \leq 1$ By use of Aizenman's 4-spin inequality [2, 3] and the Gaunt-Baker inequality [37] for the spin one-half Ising model and by using in addition Sokal's inequality [56] $\delta \geq 3$ and $\nu \geq \frac{1}{2}$ for general d. For $d \geq 5$ Aizenman and Fernández [1] have proven that $\delta = 3,\ \beta = \frac{1}{2}$. The gap index Δ is defined by

$$\chi = N^{-1} \frac{\partial^2 \ln Z}{\partial H^2}, \qquad \frac{1}{\chi} \frac{\partial^2 \chi}{\partial H^2} \propto (1 - T/T_c)^{-2\Delta} \tag{6}$$

where Z is the partition function and χ is the magnetic susceptibility. The inequality is $2\Delta \leq d\nu + \gamma$ [16, 51]. In terms of thermodynamic functions this relation is related to the renormalized coupling constant which I will discuss later. It is

$$g = -\left(\frac{v}{a^d}\right) \frac{\frac{\partial^2 \chi}{\partial H^2}}{\chi^2 \xi^d} \tag{7}$$

These inequalities are very important in shaping the structure of the equation of state at the critical point. If the inequalities are changed to equalities, then we have the behavior as predicted by scaling theory. The equalities that do depend on the spacial dimension ($\omega^* = 0$) are the hyperscaling equalities.

In simple mean field theory [63], $\gamma = 1$, $\nu = 1/2$, and $\delta = 3$ in all dimensions. $C_H = 0$ for $T > T_c$. For $T < T_c$ $C_H = 3k/2$ and so $\alpha = 0$ In mean field theory the critical index $\gamma = 1$ for the susceptibility. It behaves like $m^2/[k(T - T_c)]$ for $T > T_c$ and $m^2/[2k(T_c - T)]$ for $T < T_c$.

The Gaussian model [23] features a Gaussian distribution for each spin on a spatial lattice, rather than allowing $s = \pm 1$ as is the case for the Ising model. It has the advantage that it can be solved, but also has the disadvantage that it does not exist for $T < T_c$.

In addition there is the spherical model [23].

$$\begin{aligned} &\chi \propto t^{-2/(d-2)}, \qquad \gamma = \frac{2}{d-2}, \quad 2 < d < 4 \\ &t^{-1}|\ln t|, \qquad \gamma = 1, \qquad d = 4, \\ &t^{-1}, \qquad \gamma = 1, \qquad d > 4 \end{aligned} \tag{8}$$

where d is the spatial dimension and $t = (T/T_c - 1)$. $\nu = \gamma/2$, $d > 2$. $\delta = (d+2)/(d-2)$, $2 < d < 4$. For $d = 4$ $\delta = 3$ with logarithmic corrections. For $d > 4$, $\delta = 3$. $\beta = 1/2$, $\alpha = 0$

In this model for $T < T_c$ the spin-spin correlation decays as a power $(2-d), d > 2$ of the distance between the two points rather than the normal exponential decay. The Gaussian model, for $T > T_c$ has the same critical indices as the spherical model.

There is in addition the droplet model of Essam and Fisher [33] which has the Rushbrooke inequality as an equality.

4 Onsager Revolution

In 1941, Kramers and Wannier [44] located the exact critical point of the 2-dimensional Ising model by means of a duality transformation. In 1944 Onsager [47] gave the exact solution of the two-dimensional Ising model. This solution showed that the classical results gave incorrect results in the region of the critical point. The specific heat becomes infinite as $-\ln|T-T_c|$. In addition [68] the spontaneous magnetization vanishes like $(1-T/T_c)^{1/8}$, $T < T_c$. $\gamma = 1.75$, $\eta = 1/4$. Thus this solution differed radically from the classical results, and to make matters worse with known experimental results. It was a big problem to sort out the whys! We now know that the reason was principally due to the dimensions.

5 Series Methods

I first became interested in these statistical mechanics problems as a graduate student at UC Berkeley in the middle 1950's. I attended a special lecture course on the many body problem given by Professor Joaquin Luttinger.

When I was at the Los Alamos Scientific Laboratory there was a lot of interest in the computation of Coulomb wave functions. It turned out that continued fractions were the most efficient method of doing so. The last chapter in Wall's Book on the Analytic Theory of Continued Fractions [61] was on Padé approximants. A Padé approximant [13] is a rational fraction. It is meant to approximate an analytic function. It is denoted as $[L/M] = P_L(x)/Q_M(x)$ where $P_L(x)$ is a polynomial of degrees less than or equal to L, and $Q_M(x)$ is a polynomial of degree less than or equal to M. Some times the function name f is appended as a subscript $[L/M]_f$ where the function being approximated is not clear from the context. The formal power series $A(x) = \sum_{j=0}^{\infty} a_j x^j$ determines the coefficients by the equations

$$Q_M(x)A(x) - P_L(x) = O(x^{L+M+1}), \quad Q_M(0) = 1 \tag{9}$$

Clearly, near zero the Padé approximant will closely follow the behavior of the function being approximated. Together with John Gammel we began an investigation of their properties. In fact on all the trial cases that we tried, the results were surprisingly good. We published our first paper in 1961 [12]. The power series will converge inside a circle centered on the origin which is free from singularities. At a point near the boundary the original power series will converge to give a new convergent series about that point. Suppose that we have a function which has a singularity at $z = -1$, and all its other singularities in a circle of radius unity about $z = -2$. By appropriate manipulation of the power series all that work is not necessary. In this case the change of variables $z = 3w/(1-2w)$ maps the exterior of the circle of radius unity about $z = -2$ into the unit circle. The new series converges in the unit circle. Note that the points $z = 2$ and $z = \infty$ go into $w = 2/7$ and $w = 1/2$ respectively.

At that same time I read a paper by Domb and Sykes [32] on the use of series expansions in $w = \tanh(J/kT)$ where J equals the exchange integral. The system they studied was the ferromagnetic spin one-half Ising model. At this time it occurred to me that it would be an excellent problem to which to apply the Padé approximant method. Since the magnetic susceptibility is known to have a strong divergence at the critical point, I had the idea that the ideal way to study the critical point from the exact series would be $d\ln\chi(w)/dw$. The point is that this series would be expected to have a pole at the critical point and the residue would give us the value of γ. Padé approximants work very well on poles. I found that for three-dimensional lattices the value of $\gamma\approx 1.25$. On the other hand for the two-dimensional lattices $\gamma\approx 1.75$ as expected. These results gave a solution to the Onsager challenge. It is the case that the difference in γ found for the two-dimensional Ising model and the experimental results was due almost completely to the difference in the dimension. It was this important discovery that lead me to a year's sabbatical visiting Cyril Domb's group at Kings College, University of London. In this group, to name just a few were Michael Fisher, Martin Sykes, David Gaunt and John Essam. While in England I also began a fruitful collaboration with Stanley Rushbrooke and P.J. Wood of the University of Newcastle upon Tyne. This method of series analysis has graduated to the point that the originator is no longer mentioned!

Domb [30] says that "G.A. Baker's application of the Padé approximant (1961), a piece of mathematics which had lain dormant since the end of the nineteenth century, lead to remarkable progress, and sparked off similar applications in many other fields concerned with perturbation expansions", "the breakthrough came when Baker introduced the Padé approximant into the field". In his history of a part of the theory of critical phenomena Brush [25] has pointed out that with the Padé approximant method it was possible to obtain significant results where previous methods had failed.

6 Renormalization Group

There is instructive, homogeneity theory [31, 37, 64]. I will illustrate it with the Yang-Lee theorem [69]. Here all the singularities are on the unit circle and as the critical point is approached, the angle of the region which is free from singularities shrinks to zero. The density of singularities can be thought of as a measure and the edge is a distance $R(T)$ from the $H=0$ which is reached at $T=T_c$. It is assumed that all the important properties of the measure are taken up by the *normalization* and the *rate of approach as* $R\to 0$. The consequence of these simple homogeneity ideas leads to exponent equalities. In particular, one can get $\alpha=\alpha'$, $\gamma=\gamma'$, $\Delta=\Delta'$, $\alpha'+2\beta+\gamma=2$, and $2-\alpha'=\beta(\delta+1)$.

Another precursor to the renormalization group was the scaling theory. Kadanoff [43] studied the correlation function near the critical point. The idea was when the correlation length was long that blocks of spins behaved like one big spin because near by spins tend to be strongly aligned with each other. Therefore the spin-spin correlation function should scale somehow to give the new cell-cell correlation function. This idea gives relations between the various critical indices and restricts the number of independent ones to two. This is sometimes called 2 parameter scaling. I do not digress at this point to summarize the origins of scaling theory.

Through out the 1960's there was a lot of effort to find an exact solution for the 3-dimensional ferromagnet Ising model. Typically many people tried, but came up with models that were basically the Gaussian or spherical model. Some where around 1970 when I was at Brookhaven National Lab. Ken Wilson came to give a talk. I was tempted not to go, but since he was from Cornell, I went. Much to my surprise, what he presented was not either the Gaussian or the spherical model. I decided that this was important and got copies of his papers [65, 67]. I found the reading difficult, until I remembered that sometimes people write their papers in reverse order to how the work was done. I therefore read the paper from back to front which made the papers much easier to read. These papers served as the announcement of the renormalization group theory.

Consider an example to clarify the notions of the renormalization group theory. Let us start with a plane square lattice of Ising spins. Define a general spin $\sigma_R = \prod_{j\in R}\sigma_j$. Next define a general spin Hamiltonian

$$H_1 = \sum_A K_A^{1)}\sigma_A, \quad \sigma_i = \pm 1 \tag{10}$$

Group the spins into 3×3 cells and compute the change of variables

$$\sigma_{\text{cell}} = \sum_{\sigma_i\in\text{cell}} \sigma_i, \tag{11}$$

and require the number of spins in each cell be odd. Therefore σ_{cell} is either positive or negative. Next compute

$$\sum_{\substack{\{\sigma_i\}\\ \text{sgn (cells) fixed}}} e^{H_1} = F_2(\text{sgn}(\sigma_{\text{cells}})) \tag{12}$$

This selection is the so called "majority rule." Now one is to solve for the representation

$$F_2(\sigma_{\text{cells}}) = \exp\left[\sum_A K_A^{(2)}\sigma_A\right] \equiv \exp(H_2), \tag{13}$$

where in this case we mean

$$\sigma_A = \prod_{i\in A}[\text{sgn}(\sigma_{\text{cell }i})] \tag{14}$$

It is to be noted that by taking the logarithm of these equations, a system of non-degenerate, soluble, linear equations results for K_A. This operation thus maps $H_1 \mapsto H_2$. If the original system is translationally invariant, we can use this property to expand the system back to its original size. Thus we have a mapping of the space of N^d spin Hamiltonians onto itself. We can continue

$$\mathcal{R}^n(H_1) \mapsto H_n \tag{15}$$

It makes a map in parameter space which may tend to a fixed point.

$$\lim_{n\to\infty}\mathcal{R}^n(H_1) \stackrel{?}{=} H^* \tag{16}$$

The general picture is one of renormalization group flows to a fixed point. There may be several fixed points which are either stable or unstable. There is in addition the notion of a catchment basin one of which is associated with each fixed point. One then needs to focus one's attention on the catchment basin which is relevant to the physical system under consideration.

There are three major steps in implementing the renormalization group ideas. First, take a partial sum over the smallest scale degrees of freedom. Second readjustment of the cut-off parameter (changing the system size in our example). Third, Renormalization of the field variable to keep the scale fixed (majority rule in our case). These are the basic renormalization group operations. (It actually defines only a semi-group as information is lost as the degrees of freedom are summed out in the first step. There is no inverse mapping $\mathcal{R}^{-1}$. In addition there is no known metric so that it is impossible to rigorously tell whether the iteration process is approaching a fixed point and not a strange attractor or a limit cycle or something else unpleasant.) These mathematical points to the contrary not with standing, the method is used by most workers in the field.

There are two main methods of evaluating the results of the renormalization group. One is expand about the problem for dimension 4 [66]. The plan is to start with the Landau-Ginsburg Hamiltonian

$$\begin{aligned}\mathcal{H} = \mu_0 + \frac{1}{2}\int_{k<1/a} d\mathbf{k}(r_0 + k^2)|s_{\mathbf{k}}| \\ + u_0 \int_{k_1,k_2,k_3,k_4<1/n} s_{\mathbf{k}_1} s_{\mathbf{k}_2} s_{\mathbf{k}_3} s_{\mathbf{k}_4} \\ \times \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4\end{aligned} \tag{17}$$

The special case $u_0 = 0$ is just the Gaussian model. If we follow the procedure described above we get to leading order in r_0 and u_0 and $\epsilon = 4 - d$ all of which are assumed to be small

$$\begin{aligned}\frac{dr_0}{dl} &= 2r_0 + 48(u_0(1 - r_0)), \\ \frac{du_0}{dl} &= \epsilon u_0 - 144 u_0^2\end{aligned} \tag{18}$$

For the fixed point, the derivatives should be zero. By inspection $u_0^* = r_0^* = 0$ is a Gaussian model fixed point. There is also a non-Gaussian fixed point $u_0^* = \epsilon/144$, $r_0^* = -\epsilon/6$. This solution allows the computation of the critical exponents as functions of ϵ. Of course, the Padé approximant method can be used to accelerate the rate of convergence here. It is to be noted that this expansion, effective though it is, should be considered as a formal expansion only because Baker and Benofy [11] have shown for non-integer dimensions, these models fail to satisfy the Yang-Lee [69] theorem and so are not Ising models.

In the other method, the continuous spin Ising model is considered. The spin weight is taken to be $F(\sigma) = \exp(-\tilde{g}_0\sigma^4 - \tilde{A}\sigma^2)$ $\tilde{A}(\tilde{g}_0)$ is determined by the normalization condition,

$$\langle \sigma^2 \rangle_{\tilde{H}=0} = 1 = \frac{\int_{-\infty}^{\infty} x^2 e^{-\tilde{g}_0 x^4 - \tilde{A}x^2} dx}{\int_{-\infty}^{\infty} e^{-\tilde{g}_0 x^4 - \tilde{A}x^2} dx} \tag{19}$$

When $\tilde{g}_0 = 0$ then this continuous spin Ising model reduces to the Gaussian model. On the other hand when $\tilde{g}_0 \to \infty$ we have by the normalization condition, the regular Ising model.

The other method follows a suggestion of Parisi [48]. B. Nickel and I [17, 18] have shown how to implement that suggestion to base quantitative calculations on the perturbative calculation of the coefficients of the Callan [26]-Symanzik [57, 58] equations in fixed dimensions. This method basically involves the expansion about the Gaussian model in terms of $\tilde{g}_0$ in fixed dimension. The use of cumulants here is particularly helpful as for the Gaussian weight function all cumulants beyond second order vanish. Note is taken that the series is asymptotic. In that publication [17] we introduced the Padé Borel-Leroy method [13] which is a modification of the standard Padé method and has the feature that the known divergent aspect of the series is built in. The higher order terms diverge like $(j)!(-a)^j j^{(3+d+n)}/2$. Speaking of the 3-dimensional Ising model, this result was, I believe the first computation to make the renormalization group theory truly quantitative. There is also parallel work by [24].

We proceed to rewrite the partition function so that it looks like a lattice cut-off field theory. Namely as,

$$
\begin{aligned}
Z(H) = M^{-1}\int_{-\infty}^{\infty}\cdots\int\prod d\psi_{\mathbf{i}}\exp\Bigg\{-\frac{1}{2}\sum_{\mathbf{i}} v \\
\times\Bigg[\frac{2dZ_3}{q}\sum_{\{\delta\}}\frac{(\psi_{\mathbf{i}}-\psi_{\mathbf{i}+\delta})}{a^2} \\
+m^2 Z_3\left(\psi_{\mathbf{i}}^2-\frac{C}{Z_3}\right)+\frac{2g_0}{4!}Z_3^2\left(\psi_{\mathbf{i}}^4-\frac{6C\psi_{\mathbf{i}}^2}{Z_3}+\frac{3C^2}{Z_3^2}\right) \\
+\delta m^2 Z_3\left(\psi_{\mathbf{i}}-\frac{C}{Z_3}\right)+H_{\mathbf{i}}\psi_{\mathbf{i}}\Bigg]\Bigg\}
\end{aligned}
\tag{20}
$$

where v is the volume of the unit cell, q is the lattice coordination number and M^{-1} is a formal normalization constant. The ( ) terms are the equivalent of the field theory normal ordered products $:(Z_3\psi^2)^p:$. The $\sum_{\{\boldsymbol{\delta}\}}$ is the sum over half the nearest neighbors. Again note that if we set $g_0 = 0$ and $\delta m^2 = 0$ we get back to the Gaussian model. If we specialize to the hyper-simple cubic lattice family we get, for the two-point, inverse propagator,

$$
\Gamma^{(2)}(\mathbf{p},-\mathbf{p}) = \left\{\sum_{\mathbf{j}=\mathbf{0}}^{N-1} a^d \frac{\partial^2 \ln Z(H)}{\partial H_0 \partial H_{\mathbf{j}}}\bigg|_{H=0}\right\}^{-1} = m^2 + p^2 \tag{21}
$$

In terms of the σ variables, we have for general g_0

$$
\Gamma^{(2)}(\mathbf{p},-\mathbf{p}) = \frac{Z_3}{Ka^2}\chi^{-1}[1+\xi^2 a^2 p^2 + o(p^2)+\cdots] \tag{22}
$$

The field theory plan is to adjust Z_3 and δm^2, term by term in powers of g_0 to maintain control of the two-point inverse propagator so that the above equation will continue to hold for all g_0.

Thus we choose,

$$
m^2\xi^2 a^2 = 1, \qquad Z_3 = \frac{\chi K}{\xi^2} \propto (K_c - K)^{\eta\nu} \tag{23}
$$

We seek to approach in this way a continuum theory ($a \to 0$) through parameter adjustment in a fixed family of Hamiltonians and make our approximation by truncating the expansion in g_0 rather than by discarding the higher terms in the ϵ or the extra interactions that result from the straight forward application of Wilson's ideas. Clearly $a \to 0$ for fixed m means that $\xi \to \infty$, which means that we are approaching the critical point. Also for $d < 4$ the behavior of Z_3 forces g_0 to infinity for a fixed statistical mechanics problem. The plan is now to compute the critical phenomena in terms of the strong coupling limit $g_0 \to \infty$.

To justify this approach, some assumptions are required. Specifically, that *there exists a unique double limit* $a \to 0,\ g_0 \to \infty$. There is a technical problem. Namely the point of interest occurs at infinity. The solution is to change from the coupling constant to the renormalized coupling constant.

$$
g_R = \Gamma^{(4)}(0,0,0,0) = \frac{-\sum_{\mathbf{i},\mathbf{k},\mathbf{l}} a^{3d}\frac{\partial^4 \ln Z(H)}{\partial H_{\mathbf{0}}\partial H_{\mathbf{j}}\partial H_{\mathbf{k}}\partial H_{\mathbf{l}}}}{[a^d\frac{\partial^2 \ln Z(H)}{\partial H_{\mathbf{0}}\partial H_{\mathbf{j}}}]^4} \tag{24}
$$

This can be proven to be positive and finite in the limit as $a \to 0$ [51]. This can be computed directly by the expansion we are discussing. By Lagrange's formula on the reversion of series we can derive $g_0(g_R)$, provided that $g_R(g_0)$ is monotonic $0 \le g_0 \le \infty$. It is convenient to work with a dimensionless renormalized coupling constant

$$g = g_R m^{d-4} = -\left(\frac{v}{a^d}\right)\frac{\frac{\partial^2\chi}{\partial H^2}}{\chi^2\xi^d} \tag{25}$$

In order to locate the value of g such that $g_0 \to \infty$ one computes the Callan-Symanzik β function [26, 57, 58]:

$$\beta(g) = -(4-d)g_0\frac{\partial g}{\partial g_0}\bigg|_{a,m} = -(4-d)g + \cdots \tag{26}$$

The Callan-Symanzik analysis of the behavior of the β function indicates that the critical point is at $\beta(g^*) = 0$ with a positive derivative. If everything is a function of $g(= g^*)$ alone (the unique double limit hypothesis). By *turning the derivative* from a derivative with respect to a to a derivative with respective to g we get the result

$$\eta = \lim_{g\to g^*}\beta(g)\frac{\partial Z_3}{\partial g} \tag{27}$$

In like manner the other critical indices can also be computed. The results for the 3-dimensional Ising model are $\nu = 0.630 \pm 0.002$, $\gamma = 1.241 \pm 0.004$, $\beta = 0.324 \pm 0.006$, $\Delta = 1.566 \pm 0.006$, $\eta = 0.031 \pm 0/011$, $\delta = 4.82 \pm 0.06$, $\alpha = 0.110 \pm 0.008$, $g^* = 1.416 \pm 0.0015$ [18], but see also [17, 45] and [49].

7 Universality

Since it is hard to do these calculations, the concept of universality is often invoked. The universality hypothesis is *All critical problems may be divided into classes differentiated by* (a) *the symmetry group of the order parameter,* (b) *the dimensionality of the system, and* (c) *perhaps other criteria. Within each class the critical properties are supposed to be identical or at worst a continuous function of a very few parameters.* While in many cases universality is a valid concept, there has to date been no proof of this universality hypothesis. I give some examples to show that the situation can, in some cases, be more complex. See also Dohm [29].

First I consider the s^4 one-dimensional, continuous-spin, ferromagnetic, Ising model [8]. The continuum limit of this model has been proven to be the Ising model [41]. The spin weight is $\exp(-\tilde{A}s^2 - \tilde{g}_0 s^4)$. The special cases ($\tilde{g}_0 = 0,\ \tilde{A} = \frac{1}{2}$) and ($\tilde{g}_0 = \infty$, $\tilde{A}/\tilde{g}_0 \to -2$) are the Gaussian and Ising models respectively. Except for the case $\tilde{g}_0 = 0$ the model has its critical temperature T_c at zero. For the Ising model case the behavior is well known. In the limit as $K \to \infty$

$$\begin{aligned} &C_H/N \sim kK^2\mathrm{sech}^2(K), \qquad \chi/N \sim m^2e^{2K}/kT, \\ &\xi \sim \frac{1}{2}e^{2K^2} \end{aligned} \tag{28}$$

Using the results of Isaacson [41] and the transfer matrix method, I concluded that $C_H = kK^2/(2\tilde{g}_0)$. The specific heat vanishes at the critical point exponentially which corresponds to $\alpha = -\infty$. On the other hand, for the continuous spin Ising model, the specific heat diverges like K^2 as K goes to infinity which corresponds to $\alpha = 2$. The significant structure of this one-dimensional model, as far as the limiting field theory is concerned is identical to that of the Ising model, and leads to the same field theory for all $0 < \tilde{g}_0 \leq \infty$. However, the rate of approach is quite different since $e^{K^2} \gg e^K$. The point $g = g^*$ in the continuum field theory must necessarily be a point of non-uniform approach in the $K - \tilde{g}_0$ plane for the Callan-Symanzik functions.

Baker and Johnson [14] have investigated the question of universality in the 2-dimensional continuous-spin Ising model. The spin weight is $\exp(-\tilde{g}_0 s^4 - \tilde{A}s^2 + Hs)$. They found 5 cases. $\tilde{A}$ is computed from $\tilde{g}$ by the requirement that $\langle s^2 \rangle = 1$. These cases are separated by the behavior of $\tilde{A}$. They are (i) the Gaussian model, $\tilde{g}_0 = 0$ and $\tilde{A} = \frac{1}{2}$ (ii) $\tilde{g}_0 \neq 0$ and A less than one half, but positive, (iii) $\tilde{A} = 0$, $\tilde{g}_0 = [\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})]^2 = g_b$, (iv) $g_b < \tilde{g}_0 < \infty$, and $\tilde{A}$ negative. (v) $\tilde{g}_0 = \infty$. In cases (iv) an (v) the spin weight function has 2 peaks, and in the other cases there is only a single peak. Cases (i) and (v) are the Gaussian model and the Ising model respectively. They found, by series analysis, for case (iii) (single peak, quartic top) that $\gamma = 1.92 \pm 0.06$ for the plane square lattice and the same for the triangular lattice, but with error ± 0.03. These results are noticeably different from $\gamma = 1.75$ for case (v).

8 Value of ω^*

For the Ising model in three-dimensions, careful series analysis [6] shows that hyperscaling fails, *i.e.* $2\Delta < d\nu + \gamma$, which contradicts the renormalization group theory. The difference between the two sides of the equation was 0.019 ± 0.006. There is also some evidence of this problem found by Barma and Fisher [20, 21]. The bulk of the community sided with the renormalization group ideas. After many years (1995), the situation was finally clarified. Using a new and advanced Monte Carlo method by a very long run (about 1000 hours), Baker and Kawashima [15] established that the renormalized coupling constant, $g > 0$ which means that Fisher's anomalous dimension of the vacuum $\omega^* = 0$ so that hyperscaling holds in agreement with renormalization group theory.

9 Conformal Charge

A major advance in the theory of two-dimensional critical phenomena has been the introduction of the conformal invariance hypothesis. So far as I know, no counter-examples have been found. This idea is an extension of the scaling hypothesis, and is also called "local scale invariance." The basic papers in this area are those of Belvian et al. [22], Cardy [27], and Freidan et al. [36]. In these works it was recognized that the only possible, conformally-invariant, critical theories are broadly categorized by a parameter called the central charge. Furthermore the only possible values of this charge are

$$c = 1 - \frac{6}{m(m+1)}, \quad m = 2, 3, \ldots, \ c \geq 1 \tag{29}$$

The discovery by Zamolodchikov [70] of his c theorem and by the work of Cardy, the notion of conformal invariance has been extended beyond the critical point. Singh and Baker [53, 54] have shown that these results can be used to compute the central charge of a conformally invariant theory directly from the bulk properties by a particular hyper-universal amplitude ratio. They have used a high temperature series approach. The central charge is the algebraic center of the Virasoro algebra [38, 60] induced by conformal invariance. Wang and Baker [62] have used the Monte Carlo method to compute the central charge.

Baker et al. [19] have investigated whether the results quoted in the universality section can be conformally invariant. It would be very significant to the idea that conformal invariance is widely valid if this were not so. Specifically they study cases (ii) (single peak model), (iii) (border model) and (iv) (two peak model). They used longer series than previous studies. They studied the correlation length and the susceptibility. Their series analysis found $\gamma \sim 1.79$, and $\gamma \sim 1.024$. The best match to their results for the border model is $m = 12$ $(\Delta_1 = (6,5),\ \Delta_2 = (7,7))$. They found $\gamma = 1.7628$, $\nu = 1.0353$. For the double well case, the series are shorter (9 or 10 terms). They get $\gamma = 2.00 \pm 0.02$, $\nu = 1.08 \pm 0.01$. There is a conformal invariant theory with $m = 9$ $(\Delta_1 = (5,4),\ \Delta_2 = (5,5))$ which gives $\gamma = 2.00000$, $\nu = 1.08333$. For the single well they got ($\gamma = 1.735 \pm 0.30$, $\nu = 0.93 \pm 0.02$). For $m = 6$ $(\Delta_1 = (3,2),\ \Delta_2 = (3,3))$ gives the exponents $\gamma = 1.71428$, $\nu = 0.95238$. For each m there are various scaling dimensions $\Delta + \bar{\Delta}$ and spin dimensions $\Delta - \bar{\Delta}$.

They conclude that they have observed variations from the prediction of universality, but in some cases longer series may be required before firm conclusions can be drawn.

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References

1. Aizeman, M., Fernández, R.: J. Stat. Phys. **44**, 393 (1986)
2. Aizenman, M.: Phys. Rev. Lett. **47**, 1 (1981)
3. Aizenman, M.: Commun. Math. Phys. **86**, 1 (1982)
4. Andrews, T.: Philos. Trans. R. Soc. **159**, 575 (1869)
5. Baker, G.A. Jr.: Phys. Rev. Lett. **34**, 268 (1975)
6. Baker, G.A. Jr.: Physica **86**, 602 (1977)
7. Baker, G.A. Jr.: Phys. Rev. B **17**, 1365 (1978)
8. Baker, G.A. Jr.: Phys. Rev. Lett. **60**, 1844 (1988)
9. Baker, G.A. Jr.: Quantitative Theory of Critical Phenomena. Academic Press, Boston (1990)
10. Baker, G.A. Jr.: Phys. Rev. E **77**, 031120 (2008)
11. Baker, G.A. Jr., Benofy, L.P.: J. Stat. Phys. **29**, 699 (1982)
12. Baker, G.A. Jr., Gammel, J.L.: J. Math. Anal. Appl. **2**, 21 (1961)
13. Baker, G.A. Jr., Graves-Morris, P.R.: Padé Approximants, 2nd edn. Cambridge University Press, London (1996)
14. Baker, G.A. Jr., Johnson, J.D.: J. Phys. A **17**, L275 (1984)
15. Baker, G.A. Jr., Kawashima, N.: Phys. Rev. Lett. **75**, 994 (1995)
16. Baker, G.A. Jr., Krinsky, S.: J. Math. Phys. **18**, 590 (1977)
17. Baker, G.A. Jr., Nickel, B.G., Green, M.S., Meiron, D.I.: Phys. Rev. Lett. **36**, 1351 (1976)
18. Baker, G.A. Jr., Nickel, B.G., Meiron, D.I.: Phys. Rev. **17**, 1365 (1978)
19. Baker, G.A. Jr., Weihong, Z., Oitmaa, J.: J. Phys. **27**, 3403 (1994)
20. Barma, M., Fisher, M.E.: Phys. Rev. Lett. **54**, 2462 (1985)
21. Barma, M., Fisher, M.E.: Phys. Rev. B **31**, 5954 (1985)
22. Belvian, A.M., Polakov, M., Zamolodchikov, A.B.: Nucl. Phys. B **241**, 333 (1984)
23. Berlin, T.H., Kac, M.: Phys. Rev. **86**, 821 (1952)
24. Brézin, E., LeGuillou, J.-C., Zinn-Justin, J.: Phys. Rev. D **15**, 1544 (1977)
25. Brush, S.G.: Rev. Mod. Phys. **39**, 883 (1967)
26. Callan, C.J. Jr.: Phys. Rev. D **2**, 1541 (1970)
27. Cardy, J.L.: Nucl. Phys. B **240**, 514 (1984)
28. Curie, P.: Ann. Chim. Phys. **5**, 289 (1895)
29. Dohm, V.: Phys. Rev. E **77**, 061128 (2008)
30. Domb, C.: The Critical Point. Taylor & Francis, London (1996)
31. Domb, C., Hunter, D.L.: Proc. R. Soc. (Lond.) **86**, 1146 (1965)

32. Domb, C., Sykes, M.F.: J. Math. Phys. **2**, 63 (1961)
33. Essam, J.W., Fisher, M.E.: J. Chem. Phys. **38**, 802 (1963)
34. Fisher, M.E.: J. Math. Phys. **5**, 944 (1964)
35. Fisher, M.E.: Phys. Rev. **180**, 594 (1969)
36. Friedan, D., Qiu, Z., Shenker, S.: Phys. Rev. Lett. **52**, 1575 (1984)
37. Gaunt, D., Baker, G.A. Jr.: Phys. Rev. B **1**, 1184 (1970)
38. Gel'fand, I.M., Fuchs, D.B.: Funct. Anal. Appl. **2**, 342 (1968)
39. Glimm, J., Jaffe, A.: Commun. Math. Phys. **52**, 203 (1977)
40. Griffiths, R.B.: J. Chem. Phys. **43**, 1958 (1958)
41. Isaacson, D.: Commun. Math. Phys. **53**, 257 (1977)
42. Josephson, B.D.: Proc. Phys. Soc. (Lond.) **92**, 269–276 (1967)
43. Kadanoff, L.P.: Physics **2**, 263 (1966)
44. Kramers, H.A., Wannier, G.H.: Phys. Rev. **60**, 252–263 (1941)
45. LeGuillou, J.-C., Zinn-Justin, J.: Phys. Rev. Lett. **39**, 95 (1977)
46. Newman, C.M.: Z. Wahrscheinlichkeitstheor. Verw. Geb. (1978)
47. Onsager, L.: Phys. Rev. **65**, 117 (1944)
48. Parisi, G.: In: Brézin, E., Charap, J. M. (eds.) Proceedings of the 1973 Cargèse Summer School (private communication)
49. Pelissetto, A., Vicari, E.: arXiv:cond-mat/0012164, 10 Dec 2000
50. Rushbrooke, G.S.: J. Chem. Phys. **39**, 842 (1963)
51. Schrader, R.: Phys. Rev. B **14**, 172 (1976)
52. Simon, B.: Phys. Rev. Lett. **44**, 517 (1980)
53. Singh, R.R.P.: Physica A **177**, 123 (1991)
54. Singh, R.R.P., Baker, G.A. Jr.: Phys. Rev. Lett. **66**, 1 (1991)
55. Sokal, A.D.: J. Stat. Phys. **25**, 25–51 (1981)
56. Sokal, A.D.: Ann. Inst. Henri Poincaré **37**, 317 (1982)
57. Symanzik, K.: Commun. Math. Phys. **18**, 227 (1970)
58. Symanzik, K.: Commun. Math. Phys. **23**, 49 (1971)
59. Van der Waals, J.D.: Thesis, Leiden (1873)
60. Virasoro, M.A.: Phys. Rev. D **1**, 2933 (1970)
61. Wall, H.S.: Analytic Theory of Continued Fractions. Van Nostrand, Princeton (1948)
62. Wang, X., Baker, G.A. Jr.: J. Stat. Phys. **69**, 1069 (1992)
63. Weiss, P.: J. Phys. Radium (Paris) **6**, 667 (1907)
64. Widom, B.: J. Chem. Phys. **43**, 3998 (1965)
65. Wilson, K.G.: Phys. Rev. B **4**, 3171–3184 (1971)
66. Wilson, K.G., Fisher, M.E.: Phys. Rev. Lett. **28**, 240 (1972)
67. Wilson, K.G., Kogut, J.: Phys. Rep. C **12**, 75 (1974)
68. Yang, C.N.: Phys. Rev. **85**, 808 (1952), but derived and announced earlier by Onsager earlier without details
69. Yang, C.N., Lee, T.D.: Phys. Rev. **87**, 404–410 (1952)
70. Zamolodchikov, A.B.: In: Critical Phenomena (1983 Brasov School Conference). Progress in Physics, vol. 11, pp. 402–438. Birkhäuser, Boston (1983)